Measure Theory with Ergodic Horizons
Lecture 12

Measurable functions.

Def. Let $(X, \mathcal{I})$ and $(Y, \mathcal{J})$ be measurable spaces (i.e. $\mathcal{I}$ and $\mathcal{J}$ are σ-algebras). A function $f : X \to Y$ is said to be
(a) $(\mathcal{I}, \mathcal{J})$-measurable if $f^{-1}(J) \in \mathcal{I}$ for all $J \in \mathcal{J}$.
(b) $\mathcal{I}$-measurable (or just measurable if $\mathcal{I}$ is clear from the context) if Y is a metric space and f is $(\mathcal{I}, \mathcal{B}(Y))$-measurable.
(c) Borel if X, Y are metric spaces and f is $(\mathcal{B}(X), \mathcal{B}(Y))$-measurable.
(d) μ-measurable if μ is a measure on $\mathcal{I}$, Y is a metric space (e.g. $\mathbb{R}$) and f is Meas_μ-measurable.

Caution. Viewing $\mathbb{R}$ as both a metric space and a measure space wrt Lebesgue measure λ, a function $f : \mathbb{R} \to \mathbb{R}$ is λ-measurable $\iff$ the f-preimages of Borel sets are λ-measurable. This is weaker than demanding that f-preimages of λ-measurable sets are λ-measurable. Thus the class of λ-measurable functions is larger than the perhaps more intuitive symmetric definition would give.

Prop. Let $(X, \mathcal{I})$ and $(Y, \mathcal{J})$ be measurable spaces and $f : X \to Y$. If, for some some generating family $\mathcal{J}_0 \subseteq \mathcal{J}$, f-preimages of sets in $\mathcal{J}_0$ are in $\mathcal{I}$, then f is $(\mathcal{I}, \mathcal{J})$-measurable.

Proof. Let $\mathcal{J}' := \{J \in \mathcal{J} : f^{-1}(J) \in \mathcal{I}\}$. Then $\mathcal{J}_0 \subseteq \mathcal{J}'$ and $\mathcal{J}'$ is easily seen to be a σ-algebra because f^{-1} commutes with unions and complements, so $\mathcal{J}' = \mathcal{J}$. □

Corollary. Let $(X, \mathcal{I})$ be a measurable space, Y be a metric space, and $f : X \to Y$. If $f^{-1}(V) \in \mathcal{I}$ for all open $V \subseteq Y$, then f is $\mathcal{I}$ measurable.
In particular, if X is also a metric space, then continuous functions from X to Y

are Borel.

Proof. Just follows from the fact that open sets of Y generate $\mathcal{B}(Y)$, and also that continuous functions, by definition, are those f s.t. $f^{-1}(\text{open})$ is open. $\square$

The following is one of the reasons for building measure theory:

Theorem. Pointwise limits of measurable real-valued functions are measurable.
More generally, for any separable metric space Y and any measurable space $(X, \mathcal{I})$, if each $f_n : X \to Y$ is $\mathcal{I}$-measurable and $f(x) := \lim_{n\to\infty} f_n(x)$ for all $x \in X$, then f is $\mathcal{I}$-measurable.

Proof. By the last corollary, it suffices to check that $f^{-1}(U) \in \mathcal{I}$ for all open $U \subseteq Y$.
Note that the openness of U gives: for each $x \in X$,

$$f(x) \in U \implies \forall^\infty_{n\in\mathbb{N}}\ f_n(x) \in U,$$

but the converse doesn't hold, e.g. $U = (0,1)$ and $f_n(x) = \frac{1}{n}$. For the converse, we would need U to be closed. We can practictically achieve both via the following:

Claim. $\bigcup_{k\in\mathbb{N}} V_k = U = \bigcup_{k\in\mathbb{N}} \overline{V_k}$ for some open sets $V_k \subseteq X$.

Pf of Claim. Separability gives us a ctbl dense set $A \subseteq X$. Take

$$\mathcal{V} := \{ B_{1/n}(a) : a \in A,\ n \in \mathbb{N}^+,\ \overline{B_{1/n}(a)} \subseteq U \},$$

where $B_r(x)$ denotes the open ball of radius r centered at $x \in X$. Clearly, $\mathcal{V}$ is ctbl and $\overline{V} \subseteq U$ for each $V \in \mathcal{V}$, by definition. Conversely, for each $u \in U$, $\exists n \in \mathbb{N}^+$ with $B_{1/n}(u) \subseteq U$, and taking an $a \in B_{1/2n}(u) \cap A$, we get $u \in B_{1/2n}(a) \subseteq \overline{B_{1/2n}(a)} \subseteq B_{1/n}(u) \subseteq U$, so $u \in B_{1/2n} \in \mathcal{V}$. $\square$ (Claim)

Now we finally have: $\forall x \in X$, $f(x) \in U \iff \exists k\ \forall^\infty_n\ f_n(x) \in \overline{V_k}$. Indeed:
$\Rightarrow$: If $f(x) \in U = \bigcup_k V_k$, so $f(x) \in V_k \Rightarrow \forall^\infty_n\ f_n(x) \in V_k \subseteq \overline{V_k}$.
$\Leftarrow$: If for some $k \in \mathbb{N}$, we have $\forall^\infty_n\ f_n(x) \in \overline{V_k}$, then $f(x) := \lim_n f_n(x) \in \overline{V_k} \subseteq U$.
Thus $f^{-1}(U) := \{x \in X : f(x) = U\} = \{x \in X : \exists k\ \forall^\infty_n\ f_n(x) \in \overline{V_k}\} = \bigcup_k \bigcap_m \bigcup_{n \geq m} f_n^{-1}(\overline{V_k})$, so $f^{-1}(U)$ is $\mathcal{I}$-measurable because each $f_n^{-1}(\overline{V_k})$ is in $\mathcal{I}$. $\square$

Prop. Let X, Y be topological spaces, where Y is 2^{nd} ctbl. Let μ be a Borel measure on X that is strongly regular (e.g. X is metric space and μ is σ-finite by open sets). Let $f: X \to Y$ be a μ-measurable function. Then:

(a) f is Borel on a Borel conull set X', i.e. $f|_{X'}: X' \to Y$ is a Borel function.
(Note: $\mathcal{B}(X') = \{B \in \mathcal{B}(X) : B \subseteq X'\}$.)

(b) Luzin's Theorem. f is continuous on a closed co-ε measured set C, i.e.
$f|_C : C \to Y$ is continuous and $\mu(X \setminus C) \leq \varepsilon$.

Proof. Let $\{V_n\}$ be a ctbl basis of open sets of Y.
(a) It's enough to ensure that $f^{-1}(V_n)$ is Borel for all n. But $f^{-1}(V_n)$ is measurable so $f^{-1}(V_n) =_\mu B_n$ for some Borel set $B_n \subseteq X$. Let $Z_n \supseteq f^{-1}(V_n) \Delta B_n$ be a Borel null set and take $X' := X \setminus \bigcup_n Z_n$. Then,
$f|_{X'}^{-1}(V_n) = f^{-1}(V_n) \cap X' = B_n \cap X'$, which is a Borel subset of X' and X.

(b) It's enough to ensure that $f^{-1}(V_n)$ is open for all n. But by strong regularity, $\exists$ open U_n with $\mu(U_n \Delta f^{-1}(V_n)) \leq \varepsilon/2^{n+2}$. Then there is an open $Z_n \supseteq U_n \Delta f^{-1}(V_n)$ of measure $\leq \varepsilon/2^{n+1}$. Then let $C := X \setminus \bigcup_n Z_n$.
This C is closed, $\mu(X \setminus C) \leq \sum_n \varepsilon/2^{n+1} = \varepsilon$ and
$f|_C^{-1}(V_n) = f^{-1}(V_n) \cap C = U_n \cap C$, which open relative to C. $\square$